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2007 J. Phys. A: Math. Theor. 40 14433

(http://iopscience.iop.org/1751-8121/40/48/009)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 14433–14445 doi:10.1088/1751-8113/40/48/009

The anti-self-dual Yang–Mills equation and the Painlevé III equation

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Received 2 August 2007, in final form 16 October 2007
Published 14 November 2007
Online at stacks.iop.org/JPhysA/40/14433

Abstract
The classical transcendental solutions to the Painlevé III equation are derived from a family of solutions to the $SL(2, \mathbb{C})$ anti-self-dual Yang–Mills equation. It is also shown that the affine Weyl group symmetry of P_{III} is recovered from the symmetry of Yang's equation.

PACS numbers: 02.30.Gp, 03.50.−z, 02.30.Ik, 04.20.Jb
Mathematics Subject Classification: 33C10, 33E17, 34M55, 70S15

1. Introduction

Both the anti-self-dual Yang–Mills (ASDYM) equation and the six Painlevé equations play a key role in the theory of integrable systems. It has been shown by Mason and Woodhouse that the $SL(2, \mathbb{C})$ ASDYM equation can be reduced to the Painlevé equations under certain three-dimensional Abelian groups of conformal symmetries [1, 2].

Corrigan *et al* have constructed a family of solutions to Yang's equation, which is equivalent to the ASDYM equation, in the case of $SL(2, \mathbb{C})$ [3, 4]. These solutions can be expressed in terms of Hankel determinants whose entries satisfy the Laplace equation. On the other hand, it is known that the classical transcendental solutions to the Painlevé equations can be expressed in terms of some determinants whose entries satisfy (confluent) hypergeometric differential equations [5]. In the previous paper [6], the author has presented the explicit expression for a family of solutions to Yang's equation that corresponds to the classical transcendental solutions of the Painlevé II and IV equations. Shah and Woodhouse have constructed a similar expression for the Painlevé VI equation [7].

One of the aims of this paper is to present a similar expression in terms of τ-functions for a family of classical transcendental solutions to the Painlevé III equation (P_{III})

$$\frac{d^2 y_*}{d\rho^2} = \frac{1}{y_*}\left(\frac{dy_*}{d\rho}\right)^2 - \frac{1}{\rho}\frac{dy_*}{d\rho} - \frac{4}{\rho}[\eta_\infty\theta_\infty y_*^2 + \eta_0(\theta_0+1)] + 4\eta_\infty^2 y_*^3 - \frac{4\eta_0^2}{y_*}, \tag{1.1}$$

or its equivalent

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{y^2}{t^2}\left(\eta_\infty^2 y - \eta_\infty\theta_\infty\right) - \frac{\eta_0(\theta_0+1)}{t} - \frac{\eta_0^2}{y}, \tag{1.2}$$

which is called Painlevé III' equation. These two equations are mutually connected through the variable transformations $t = \rho^2$, $y = \rho y_*$.

It is also known that each of the Painlevé equations, except for P_I, admits the affine Weyl group symmetry as the group of Bäcklund transformations. It is meaningful to explain such a symmetry from that of the ASDYM equation. Another purpose of this paper is to derive the (extended) affine Weyl group symmetry of $P_{III'}$ from the symmetry of Yang's equation.

In section 2, we give a brief review of Yang's equation, its symmetry and a family of special solutions. We summarize the derivation of $P_{III'}$ from the ASDYM equation in section 3. In section 4, we show that the classical transcendental solutions of P_{III} can be derived from the family of solutions to Yang's equation. In section 5, we mention the derivation of $P_{III'}$ from Yang's equation and show that all the Bäcklund transformations of $P_{III'}$ can be recovered from those of Yang's equation. Appendices A and B are devoted to a remark on the relationship to the Ernst equation.

2. Yang's equation, its symmetry and determinant solutions

In this section we give a brief review of Yang's equation [8], its symmetries and a family of special solutions expressed in terms of Hankel determinants [3, 4].

The $SL(2,\mathbb{C})$ ASDYM equation is given by

$$\begin{aligned} &\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0,\\ &\partial_{\tilde z} A_{\tilde w} - \partial_{\tilde w} A_{\tilde z} + [A_{\tilde z}, A_{\tilde w}] = 0,\\ &\partial_z A_{\tilde z} - \partial_{\tilde z} A_z - \partial_w A_{\tilde w} + \partial_{\tilde w} A_w + [A_z, A_{\tilde z}] - [A_w, A_{\tilde w}] = 0, \end{aligned} \tag{2.1}$$

where A_z, A_w, $A_{\tilde z}$ and $A_{\tilde w}$ are the components of the gauge potential $A = A_z\,dz + A_w\,dw + A_{\tilde z}\,d\tilde z + A_{\tilde w}\,d\tilde w$ and are $\mathfrak{sl}(2,\mathbb{C})$-valued functions. The first two equations of (2.1) are the local integrability conditions for the existence of two matrix-valued functions H and $\widetilde H$ such that

$$\partial_z H = -A_z H, \qquad \partial_w H = -A_w H, \qquad \partial_{\tilde z}\widetilde H = -A_{\tilde z}\widetilde H, \qquad \partial_{\tilde w}\widetilde H = -A_{\tilde w}\widetilde H. \tag{2.2}$$

They are determined uniquely by A up to $H \mapsto H\widetilde M$, $\widetilde H \mapsto \widetilde H M$, where M depends only on z and w, and $\widetilde M$ depends only on $\tilde z$ and $\tilde w$. The third equation of (2.1) holds if and only if the J*-matrix* defined by $J = \widetilde H^{-1} H$ satisfies Yang's equation

$$\partial_w(J^{-1}\partial_{\tilde w} J) - \partial_z(J^{-1}\partial_{\tilde z} J) = 0. \tag{2.3}$$

It is obvious that Yang's equation is invariant under the transformation

$$J \mapsto M^{-1} J \widetilde M, \tag{2.4}$$

which means that one can regard the transformation (2.4) as the Bäcklund transformation of Yang's equation (2.3). It is known that Yang's equation (2.3) also admits another Bäcklund transformation. We set

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2 + eg \end{pmatrix} \tag{2.5}$$

to express Yang's equation (2.3) as the coupled nonlinear equations

$$\begin{aligned}
&\partial_z\partial_{\tilde z}(\log f) + \frac{(\partial_{\tilde z}e)(\partial_z g)}{f^2} = \partial_w\partial_{\tilde w}(\log f) + \frac{(\partial_{\tilde w}e)(\partial_w g)}{f^2},\\
&\partial_{\tilde z}\left(\frac{\partial_z g}{f^2}\right) = \partial_{\tilde w}\left(\frac{\partial_w g}{f^2}\right),\\
&\partial_z\left(\frac{\partial_{\tilde z} e}{f^2}\right) = \partial_w\left(\frac{\partial_{\tilde w} e}{f^2}\right).
\end{aligned}\tag{2.6}$$

Lemma 2.1. *Let (e, f, g) be a solution to (2.6). Then $(\hat e, \hat f, \hat g)$ defined by*

$$\hat f = \frac{1}{f}, \qquad \partial_z\hat g = \frac{\partial_{\tilde w}e}{f^2}, \qquad \partial_w\hat g = \frac{\partial_{\tilde z}e}{f^2}, \qquad \partial_{\tilde z}\hat e = \frac{\partial_w g}{f^2}, \qquad \partial_{\tilde w}\hat e = \frac{\partial_z g}{f^2} \tag{2.7}$$

is also a solution.

We call (2.7) the transformation β. Obviously we have $\beta^2 = 1$.

As a particular example of the Bäcklund transformation (2.4), we consider the transformation γ defined by

$$\gamma: \qquad J \mapsto \begin{pmatrix} & 1\\ 1 & \end{pmatrix} J \begin{pmatrix} & 1\\ 1 & \end{pmatrix}. \tag{2.8}$$

We also have $\gamma^2 = 1$. Since $\beta\gamma \neq \gamma\beta$, it is possible to generate solutions to Yang's equation by operating one after the other. In fact, Corrigan *et al* [3, 4] have constructed a family of solutions to Yang's equation by such a procedure.

Proposition 2.2 [3, 4]**.** *Define the functions $\tau_n^m (m \in \mathbb{Z}, n \in \mathbb{Z}_{\geqslant 0})$ by*

$$\tau_n^m = \begin{vmatrix} \varphi_{m-n+1} & \varphi_{m-n+2} & \cdots & \varphi_m\\ \varphi_{m-n+2} & \varphi_{m-n+3} & \cdots & \varphi_{m+1}\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_m & \varphi_{m+1} & \cdots & \varphi_{m+n-1} \end{vmatrix}, \tag{2.9}$$

where the entries φ_j satisfy

$$\partial_{\tilde w}\varphi_j = \partial_z\varphi_{j+1}, \qquad \partial_{\tilde z}\varphi_j = \partial_w\varphi_{j+1} \tag{2.10}$$

and the Laplace equation

$$(\partial_w\partial_{\tilde w} - \partial_z\partial_{\tilde z})\,\varphi_j = 0. \tag{2.11}$$

Then

$$J = \frac{1}{\tau_n^m}\begin{pmatrix} \tau_n^{m-1} & \tau_{n+1}^m\\ \tau_{n-1}^m & \tau_n^{m+1} \end{pmatrix} \tag{2.12}$$

gives rise to a family of solutions to Yang's equation (2.3).

We remark that the functions τ_n^m defined by (2.9) satisfy the bilinear relations [9]

$$\begin{aligned}
&D_{\tilde w}\tau_n^m \cdot \tau_{n-1}^{m+1} = D_z\tau_n^{m+1}\cdot\tau_{n-1}^m,\\
&D_{\tilde z}\tau_n^m \cdot \tau_{n-1}^{m+1} = D_w\tau_n^{m+1}\cdot\tau_{n-1}^m,\\
&\tau_{n+1}^m\tau_{n-1}^m = \tau_n^{m+1}\tau_n^{m-1} - \tau_n^m\tau_n^m,
\end{aligned}\tag{2.13}$$

where $D_{\tilde w}$, D_z, $D_{\tilde z}$ and D_w are Hirota's bilinear operators.

3. Reduction to the Painlevé III' equation

Mason and Woodhouse have shown that the $SL(2,\mathbb{C})$ ASDYM equation can be reduced to the Painlevé equations [1, 2]. Suppose that the gauge potential A is invariant under the action of the Jordan group of degree 4. Then the ASDYM equation is reduced to a system of ordinary differential equations, from which one can derive the Painlevé equation.

Let us summarize the derivation of the Painlevé III' equation from the ASDYM equation according to [1, 2]. In the case of $\mathrm{P_{III'}}$, we take the Jordan group of the form

$$\begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & b & c \\ & & & b \end{pmatrix}, \tag{3.1}$$

and the above criterion leads us to the coordinate transformation

$$p = z, \qquad q = \tilde{z}, \qquad r = \log\tilde{w}, \qquad t = w\tilde{w}. \tag{3.2}$$

We rewrite the gauge potential A in the form $A = P\,\mathrm{d}p + Q\,\mathrm{d}q + R\,\mathrm{d}r + T\,\mathrm{d}t$, where P, Q, R and T depend only on t. Since it is possible to fix $T = 0$ by a gauge transformation, we have

$$A_z = P, \qquad A_{\tilde{z}} = Q, \qquad A_w = 0, \qquad A_{\tilde{w}} = \mathrm{e}^{-r}R. \tag{3.3}$$

Substituting the result into the ASDYM equation (2.1), we get a system of ordinary differential equations

$$P' = 0, \qquad Q' = [Q, R], \qquad R' = t[P, Q], \qquad ' = t\frac{\mathrm{d}}{\mathrm{d}t}. \tag{3.4}$$

The residual gauge freedom can be exploited to reduce P to the form $P = \mathrm{diag}(k, -k)(k \neq 0)$, when the eigenvalue of P is a non-zero constant. We then obtain for six unknowns a system of equations

$$\begin{aligned} &R'_{11} = 0, \qquad R'_{12} = 2ktQ_{12}, \qquad R'_{21} = -2ktQ_{21}, \\ &Q'_{11} = Q_{12}R_{21} - Q_{21}R_{12}, \\ &Q'_{12} = 2(Q_{11}R_{12} - Q_{12}R_{11}), \\ &Q'_{21} = 2(Q_{21}R_{11} - Q_{11}R_{21}). \end{aligned} \tag{3.5}$$

We find that the quantities

$$\begin{aligned} &l^2 = \tfrac{1}{2}\mathrm{tr}(Q^2) = Q_{11}^2 + Q_{12}Q_{21}, \\ &m = \mathrm{tr}(PR) = 2kR_{11}, \\ &n = \mathrm{tr}(QR) = 2Q_{11}R_{11} + Q_{12}R_{21} + Q_{21}R_{12} \end{aligned} \tag{3.6}$$

are the first integrals and that the system (3.5) essentially has three unknown variables.

By using (3.5) and (3.6) we obtain for $y = R_{12}/Q_{12}$ and $x = -Q_{11} + l$ a system of equations

$$\begin{aligned} &y' = 2y^2x - (\eta_\infty y^2 + \theta_0 y + \eta_0 t), \\ &x' = -2yx^2 + (2\eta_\infty y + \theta_0)x - \tfrac{1}{2}\eta_\infty(\theta_0 + \theta_\infty), \end{aligned} \tag{3.7}$$

with

$$\eta_0 = -2k, \qquad \eta_\infty = 2l, \qquad \theta_0 = -\frac{m}{k}, \qquad \theta_\infty = \frac{n}{l}, \tag{3.8}$$

which are precisely the Hamiltonian system for $\mathrm{P_{III'}}$ (1.2). Note that one can get for $\hat{y} = R_{21}/Q_{21}$ the equation $\mathrm{P_{III'}}$ with the parameter θ_0 being replaced by $\theta_0 - 2$.

4. The J-matrix for the classical transcendental solutions to $\mathrm{P_{III}}$

In this section we construct the J-matrix for the classical transcendental solutions to the Hamiltonian system for the Painlevé III equation,

$$\begin{aligned} y_*' &= 4y_*^2x_* - 2\eta_\infty\rho y_*^2 - (2\theta_0+1)y_* - 2\eta_0\rho, \\ x_*' &= -4y_*x_*^2 + (4\eta_\infty\rho y_* + 2\theta_0+1)x_* - \eta_\infty(\theta_0+\theta_\infty)\rho, \end{aligned} \tag{4.1}$$

where we denote $' = \rho\frac{\mathrm{d}}{\mathrm{d}\rho}$. This system can be obtained from (3.7) by the variable transformations $t=\rho^2$, $y=\rho y_*$ and $x=\rho^{-1}x_*$.

Proposition 4.2 [5, 10]**.** *Define the functions τ_n^ν $(n\in\mathbb{Z}_{\geqslant 0}, \nu\in\mathbb{C})$ by*

$$\tau_n^\nu = \begin{vmatrix} \psi_\nu^{(0)} & \psi_\nu^{(1)} & \cdots & \psi_\nu^{(n-1)} \\ \psi_\nu^{(1)} & \psi_\nu^{(2)} & \cdots & \psi_\nu^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_\nu^{(n-1)} & \psi_\nu^{(n)} & \cdots & \psi_\nu^{(2n-2)} \end{vmatrix}, \qquad \psi_\nu^{(k)} = \left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^k\psi_\nu, \tag{4.2}$$

where ψ_ν is given in terms of (modified) Bessel functions by

$$\psi_\nu = \begin{cases} c_1J_\nu + c_2Y_\nu, & (4\eta_0\eta_\infty = +1) \\ c_1I_\nu + c_2I_{-\nu}, & (4\eta_0\eta_\infty = -1) \end{cases} \tag{4.3}$$

with c_1 and c_2 being arbitrary complex constants. Then

$$y_* = \frac{1}{2\eta_\infty}\frac{\tau_{n+1}^\nu\tau_n^{\nu+1}}{\tau_{n+1}^{\nu+1}\tau_n^\nu}, \qquad x_* = -\frac{1}{4\eta_0\rho}\frac{\tau_{n+1}^{\nu+1}\tau_{n-1}^{\nu+1}}{\tau_n^{\nu+1}\tau_n^{\nu+1}} \tag{4.4}$$

with

$$\theta_\infty = \nu+n+1, \qquad \theta_0 = -\nu+n-1 \tag{4.5}$$

give rise to a family of classical transcendental solutions to the Hamiltonian system for P_{III}.

Note that we have for the τ-functions (4.2) the following bilinear relations:

$$\begin{aligned} &\tau_{n+1}^\nu\tau_{n-1}^\nu = 4\eta_0\eta_\infty\rho^2\left(\tau_n^{\nu+1}\tau_n^{\nu-1} - \tau_n^\nu\tau_n^\nu\right), \\ &(4\eta_0\eta_\infty)^{-1}\tau_{n+1}^\nu\tau_{n-1}^{\nu+1} = \tau_{n+1}^{\nu+1}\tau_{n-1}^\nu - 2n\rho\tau_n^{\nu+1}\tau_n^\nu, \\ &\rho\tau_{n+1}^{\nu-1}\tau_n^{\nu+1} = -4\eta_0\eta_\infty\rho\tau_{n+1}^{\nu+1}\tau_n^{\nu-1} + 2\nu\tau_{n+1}^\nu\tau_n^\nu, \\ &(\rho D_\rho - \nu - n)\tau_{n+1}^\nu\cdot\tau_n^{\nu+1} = -4\eta_0\eta_\infty\rho\tau_{n+1}^{\nu+1}\tau_n^\nu, \end{aligned} \tag{4.6}$$

where D_ρ is Hirota's bilinear operator. By using these relations and (3.6), we see that the matrices Q and R can be calculated as

$$Q = \begin{pmatrix} \dfrac{\eta_\infty}{2} + \dfrac{1}{4\eta_0\rho^2}\dfrac{\tau_{n+1}^{\nu+1}\tau_{n-1}^{\nu+1}}{\tau_n^{\nu+1}\tau_n^{\nu+1}} & C_n\eta_\infty\rho^{-(\nu+n+1)}\dfrac{\tau_{n+1}^{\nu+1}\tau_n^\nu}{\tau_n^{\nu+1}\tau_n^{\nu+1}} \\ -C_n^{-1}\dfrac{1}{4\eta_0}\rho^{\nu+n-1}\dfrac{\tau_n^{\nu+2}\tau_{n-1}^{\nu+1}}{\tau_n^{\nu+1}\tau_n^{\nu+1}} & -\dfrac{\eta_\infty}{2} - \dfrac{1}{4\eta_0\rho^2}\dfrac{\tau_{n+1}^{\nu+1}\tau_{n-1}^{\nu+1}}{\tau_n^{\nu+1}\tau_n^{\nu+1}} \end{pmatrix}, \tag{4.7}$$

and

$$R = \begin{pmatrix} \dfrac{\nu+1-n}{2} & \dfrac{1}{2}C_n\rho^{-(\nu+n)}\dfrac{\tau_{n+1}^\nu}{\tau_n^{\nu+1}} \\ -C_n^{-1}\dfrac{1}{8\eta_0\eta_\infty}\rho^{\nu+n}\dfrac{\tau_{n-1}^{\nu+2}}{\tau_n^{\nu+1}} & -\dfrac{\nu+1-n}{2} \end{pmatrix}, \tag{4.8}$$

respectively, with $C_n = (-2\eta_0)^{-2n}(-2\eta_\infty)^{-n}$.

Let us solve the linear equations (2.10) and (2.11) in the coordinate system (p, q, r, t) given by (3.2). Then we get

$$\varphi_j = (-2\eta_0)^{-j}\,\mathrm{e}^{-\eta_0 z+\eta_\infty \tilde{z}}\,\tilde{w}^{\nu+1-j}\rho^{-(\nu+1-j)}\psi_{\nu+1-j}, \tag{4.9}$$

where ψ_ν is given by (4.3). Equations (2.10) and (2.11) are reduced to the contiguity relations

$$\psi_\nu' - \nu\psi_\nu = -4\eta_0\eta_\infty\rho\psi_{\nu+1}, \qquad \psi_\nu' + \nu\psi_\nu = \rho\psi_{\nu-1}, \tag{4.10}$$

and Bessel's differential equation $\psi_\nu'' + (4\eta_0\eta_\infty\rho^2 - \nu^2)\psi_\nu = 0$, respectively.

Theorem 4.2. *Define a sequence of functions φ_j $(j \in \mathbb{Z})$ by*

$$\varphi_j = (-2\eta_0)^{-j}\,\mathrm{e}^{-\eta_0 z+\eta_\infty \tilde{z}}\,\tilde{w}^{\nu+1-j}\rho^{-(\nu+1-j)}\psi_{\nu+1-j}, \tag{4.11}$$

and the functions τ_n^m by (2.9). Then the J-matrix corresponding to the classical transcendental solutions of the Painlevé III equation is given by

$$J = \frac{1}{\tau_n^0}\begin{pmatrix} \tau_n^{-1} & \tau_{n+1}^0 \\ \tau_{n-1}^0 & \tau_n^1 \end{pmatrix}. \tag{4.12}$$

We find that the components of the gauge potential are recovered by

$$\begin{aligned} A_z &= -\partial_z H H^{-1}, & A_w &= -\partial_w H H^{-1}, \\ A_{\tilde{z}} &= (-\partial_{\tilde{z}} H + H J^{-1}\partial_{\tilde{z}} J)H^{-1}, & A_{\tilde{w}} &= (-\partial_{\tilde{w}} H + H J^{-1}\partial_{\tilde{w}} J)H^{-1} \end{aligned} \tag{4.13}$$

with

$$H = \begin{pmatrix} \tilde{w}^{-\frac{1}{2}(\nu+1)}\,\mathrm{e}^{\frac{1}{2}(\eta_0 z-\eta_\infty\tilde{z})} & \\ & \tilde{w}^{\frac{1}{2}(\nu+1)}\,\mathrm{e}^{-\frac{1}{2}(\eta_0 z-\eta_\infty\tilde{z})} \end{pmatrix}. \tag{4.14}$$

Proof. What we must prove is the following six relations:

$$\frac{\left(\partial_{\tilde{z}}\tau_n^{m-1}\right)\tau_n^{m+1} - \left(\partial_{\tilde{z}}\tau_{n-1}^m\right)\tau_{n+1}^m - \left(\partial_{\tilde{z}}\tau_n^m\right)\tau_n^m}{\left(\tau_n^m\right)^2} = \frac{1}{4\eta_0\rho^2}\frac{\tau_{n+1}^{\nu+1-m}\tau_{n-1}^{\nu+1-m}}{\left(\tau_n^{\nu+1-m}\right)^2}, \tag{4.15}$$

$$\frac{D_{\tilde{z}}\tau_{n+1}^m\cdot\tau_n^{m+1}}{\left(\tau_n^m\right)^2} = (-2\eta_0)^{-m-n}(4\eta_0\eta_\infty)^{-n}\,\mathrm{e}^{-\eta_0 z+\eta_\infty\tilde{z}}\,\tilde{w}^{\nu+1-m-n} \times \eta_\infty\rho^{-(\nu+1-m+n)}\frac{\tau_{n+1}^{\nu+1-m}\tau_n^{\nu-m}}{\left(\tau_n^{\nu+1-m}\right)^2}, \tag{4.16}$$

$$\frac{D_{\tilde{z}}\tau_n^{m-1}\cdot\tau_{n-1}^m}{\left(\tau_n^m\right)^2} = (-2\eta_0)^{m+n}(4\eta_0\eta_\infty)^n\,\mathrm{e}^{\eta_0 z-\eta_\infty\tilde{z}}\,\tilde{w}^{-(\nu+1-m-n)} \times \frac{1}{4\eta_0}\rho^{\nu-1-m+n}\frac{\tau_n^{\nu+2-m}\tau_{n-1}^{\nu+1-m}}{\left(\tau_n^{\nu+1-m}\right)^2}, \tag{4.17}$$

and

$$\left(\partial_{\tilde{w}}\tau_n^{m+1}\right)\tau_n^{m-1} - \left(\partial_{\tilde{w}}\tau_{n+1}^m\right)\tau_{n-1}^m - \left(\partial_{\tilde{w}}\tau_n^m\right)\tau_n^m = 0, \tag{4.18}$$

$$\frac{D_{\tilde{w}}\tau_{n+1}^m\cdot\tau_n^{m+1}}{\left(\tau_n^m\right)^2} = \frac{1}{2}(-2\eta_0)^{-(m+n)}(4\eta_0\eta_\infty)^{-n}\,\mathrm{e}^{-\eta_0 z+\eta_\infty\tilde{z}}\,\tilde{w}^{\nu-m-n}\rho^{-\nu+m-n}\frac{\tau_{n+1}^{\nu-m}}{\tau_n^{\nu+1-m}}, \tag{4.19}$$

$$\frac{D_{\tilde{w}}\tau_{n-1}^m\cdot\tau_n^{m-1}}{\left(\tau_n^m\right)^2} = -(-2\eta_0)^{m+n}(4\eta_0\eta_\infty)^n\,\mathrm{e}^{\eta_0 z-\eta_\infty\tilde{z}}\,\tilde{w}^{-\nu+m+n-2}\rho^{\nu-m+n}\frac{1}{8\eta_0\eta_\infty}\frac{\tau_{n-1}^{\nu+2-m}}{\tau_n^{\nu+1-m}}. \tag{4.20}$$

From the linear relation $\partial_{\tilde{z}}\varphi_j = \eta_\infty\varphi_j$ and the third relation of (2.13), we have

$$\left(\partial_{\tilde{z}}\tau_n^{m-1}\right)\tau_n^{m+1} - \left(\partial_{\tilde{z}}\tau_{n-1}^{m}\right)\tau_{n+1}^{m} - \left(\partial_{\tilde{z}}\tau_n^{m}\right)\tau_n^{m} = \eta_\infty\tau_{n+1}^{m}\tau_{n-1}^{m}. \tag{4.21}$$

Since it is easy to see that

$$\tau_n^m = (-2\eta_0)^{-mn}\,\mathrm{e}^{n(-\eta_0 z+\eta_\infty\tilde{z})}\,\tilde{w}^{(\nu+1-m)n}\rho^{-(\nu+1-m)n}\left(4\eta_0\eta_\infty\rho^2\right)^{\binom{n}{2}}\tau_n^{\nu+1-m}, \tag{4.22}$$

we get the relation (4.15). In a similar way, we can verify (4.16) and (4.17).

By using $\partial_{\tilde{w}}\varphi_j = -\eta_0\varphi_{j+1}$, we have

$$\begin{aligned}
\partial_{\tilde{w}}\tau_n^{m+1} &= -\eta_0|m-n+2,\ldots,m,m+2|,\\
\partial_{\tilde{w}}\tau_{n+1}^{m} &= -\eta_0|m-n,\ldots,m-1,m+1|,\\
\partial_{\tilde{w}}\tau_n^{m} &= -\eta_0|m-n+1,\ldots,m-1,m+1|,
\end{aligned} \tag{4.23}$$

where we denote

$$j = \begin{pmatrix}\varphi_j\\ \varphi_{j+1}\\ \vdots\end{pmatrix}. \tag{4.24}$$

Set $D := |m-n,\ldots,m-1,m+1|$. Then, the bilinear relation (4.18) is reduced to Jacobi's identity

$$D\cdot D\begin{bmatrix}1 & n+1\\ 1 & n+1\end{bmatrix} = D\begin{bmatrix}1\\1\end{bmatrix}D\begin{bmatrix}n+1\\n+1\end{bmatrix} - D\begin{bmatrix}1\\n+1\end{bmatrix}D\begin{bmatrix}n+1\\1\end{bmatrix}, \tag{4.25}$$

where $D\left[\begin{smallmatrix}i_1 & \cdots & i_k\\ j_1 & \cdots & j_k\end{smallmatrix}\right]$ is the minor obtained by deleting the rows with indices $i_1,\ldots,i_k$ and the columns with indices $j_1,\ldots,j_k$.

From the first bilinear relation of (2.13), we have

$$\frac{D_{\tilde{w}}\tau_{n+1}^{m}\cdot\tau_n^{m+1}}{\left(\tau_n^m\right)^2} = \partial_z\left(\frac{\tau_{n+1}^{m+1}}{\tau_n^m}\right), \tag{4.26}$$

which yields the relation (4.19) due to (4.22). In a similar way, we get (4.20). □

5. Symmetry of the Painlevé III' equation

It is known that the Painlevé equations admit the affine Weyl group symmetry as the groups of Bäcklund transformations. In this section, we derive $\mathrm{P_{III'}}$ directly from Yang's equation and show that the (extended) affine Weyl group symmetry of $\mathrm{P_{III'}}$ is recovered from the symmetry of Yang's equation.

5.1. From Yang's equation to $P_{\mathrm{III'}}$

Let us specialize the J-matrix as

$$J = \begin{pmatrix}\mathrm{e}^{\frac{1}{2}(\eta_0 p+\eta_\infty q)+\theta_+ r}A & \mathrm{e}^{\frac{1}{2}(-\eta_0 p+\eta_\infty q)+\theta_- r}B\\ \mathrm{e}^{\frac{1}{2}(\eta_0 p-\eta_\infty q)-\theta_- r}C & \mathrm{e}^{-\frac{1}{2}(\eta_0 p+\eta_\infty q)-\theta_+ r}D\end{pmatrix}, \tag{5.1}$$

where η_0, η_∞ and $\theta_\pm$ are certain constants[1], and A, B, C and D are functions of t satisfying $AD - BC = 1$. Then, Yang's equation (2.3) can be expressed by

$$\begin{aligned}
&[A'D - BC' + \theta_+ AD + \theta_- BC]' = 0,\\
&[AD' - B'C - \theta_+ AD - \theta_- BC]' = 0,\\
&[B'D - BD' + (\theta_+ + \theta_-)BD]' = -\eta_0\eta_\infty t BD,\\
&[AC' - A'C - (\theta_+ + \theta_-)AC]' = -\eta_0\eta_\infty t AC,
\end{aligned} \tag{5.2}$$

where we denote $' = t\frac{\mathrm{d}}{\mathrm{d}t}$. Since one can see that $B'C - BC' + (\theta_+ + \theta_-)BC$ and $A'D - AD' + (\theta_+ + \theta_-)AD$ are constants, we introduce the parameters θ_0 and θ_∞ by

$$\begin{aligned}
&B'C - BC' + (\theta_+ + \theta_-)BC = -\tfrac{1}{2}(\theta_0 + \theta_\infty),\\
&A'D - AD' + (\theta_+ + \theta_-)AD = -\tfrac{1}{2}(\theta_0 - \theta_\infty).
\end{aligned} \tag{5.3}$$

Let us introduce the variables X and Y by

$$X = BC = AD - 1, \qquad Y = \frac{B'}{B} - \frac{D'}{D} + (\theta_+ + \theta_-). \tag{5.4}$$

From (5.3) and the definition of X, we have

$$\begin{aligned}
&\frac{A'}{A} - \frac{D'}{D} = \frac{-\frac{1}{2}(\theta_0 - \theta_\infty)}{1 + X} - (\theta_+ + \theta_-),\\
&\frac{A'}{A} + \frac{D'}{D} = \frac{X'}{1 + X},
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
&\frac{B'}{B} - \frac{C'}{C} = \frac{-\frac{1}{2}(\theta_0 + \theta_\infty)}{X} - (\theta_+ + \theta_-),\\
&\frac{B'}{B} + \frac{C'}{C} = \frac{X'}{X}.
\end{aligned} \tag{5.6}$$

Then, we obtain from the definition of Y

$$2Y = \frac{X'}{X} + \frac{-\frac{1}{2}(\theta_0 + \theta_\infty)}{X} - \frac{X'}{1 + X} + \frac{-\frac{1}{2}(\theta_0 - \theta_\infty)}{1 + X}, \tag{5.7}$$

namely,

$$X' = 2YX(1 + X) + \theta_0 X + \tfrac{1}{2}(\theta_0 + \theta_\infty). \tag{5.8}$$

On the other hand, we get

$$\begin{aligned}
\frac{B'}{B} + \frac{D'}{D} &= \frac{1}{2}\left[\frac{X'}{X} + \frac{-\frac{1}{2}(\theta_0 + \theta_\infty)}{X} + \frac{X'}{1 + X} - \frac{-\frac{1}{2}(\theta_0 - \theta_\infty)}{1 + X}\right]\\
&= 2YX + Y + \theta_0,
\end{aligned} \tag{5.9}$$

by using (5.8). Then the third equation of (5.2) yields

$$Y' + Y(2YX + Y + \theta_0) = -\eta_0\eta_\infty t. \tag{5.10}$$

It is easy to see that equations (5.8) and (5.10) are reduced to the Hamiltonian system for $P_{\mathrm{III}'}$ (3.7) by the variable transformations $Y = \eta_\infty y$ and $X = -\eta_\infty^{-1} x$.

[1] We see later that it is possible to put $\theta_\pm = (\theta_\infty \pm \theta_0)/2$.

5.2. Bäcklund transformations

It is obvious that Yang's equation (2.3) is invariant under the scaling transformations

$$z \mapsto \lambda z, \qquad \tilde{z} \mapsto \lambda^{-1}\tilde{z} \tag{5.11}$$

and

$$z \mapsto \mu^{-1}z, \qquad w \mapsto \mu^{-1}w. \tag{5.12}$$

Each of them induces for $P_{III'}$ the transformations

$$\eta_0 \mapsto \lambda^{-1}\eta_0, \qquad \eta_\infty \mapsto \lambda\eta_\infty, \qquad y \mapsto \lambda^{-1}y, \qquad x \mapsto \lambda x, \tag{5.13}$$

and

$$t \mapsto \mu^{-1}t, \qquad \eta_0 \mapsto \mu\eta_0, \tag{5.14}$$

respectively. We denote their composition by $\psi(\lambda, \mu)$ according to [10].

Next, we consider the following Bäcklund transformation:

$$\gamma_2: \qquad J \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} J \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \tag{5.15}$$

of Yang's equation. It is easy to see that the action of γ_2 is reduced to

$$\begin{aligned} \gamma_2\colon \quad & \eta_\infty \mapsto -\eta_\infty, \qquad \theta_+ \mapsto -\theta_-, \qquad \theta_- \mapsto -\theta_+, \\ & A \mapsto C, \quad B \mapsto -D, \qquad C \mapsto A, \qquad D \mapsto -B, \end{aligned} \tag{5.16}$$

under the specialization of (5.1). Defining the transformation s_2 by $s_2 = \psi(-1, -1)\gamma_2$, we get

$$s_2\colon \quad \theta_\infty \mapsto -\theta_\infty, \qquad y \mapsto -y, \qquad x \mapsto \eta_\infty - x, \qquad t \mapsto -t. \tag{5.17}$$

We also find that Yang's equation is invariant under the transformation

$$\chi\colon \quad z \leftrightarrow \tilde{z}, \qquad w \leftrightarrow \tilde{w}, \qquad J \mapsto {}^tJ. \tag{5.18}$$

The action of χ is reduced to

$$\begin{aligned} \chi\colon \quad & \eta_0 \mapsto \eta_\infty, \qquad \eta_\infty \mapsto \eta_0, \qquad \theta_+ \mapsto -\theta_+, \\ & A \mapsto t^{\theta_+}A, \qquad B \mapsto t^{-\theta_-}C, \quad C \mapsto t^{\theta_-}B, \qquad D \mapsto t^{-\theta_+}D. \end{aligned} \tag{5.19}$$

Introducing the transformation s_1 by $s_1 = s_2\psi(\eta_\infty/\eta_0, 1)\chi s_2$, we have

$$s_1\colon \quad \theta_0 \leftrightarrow \theta_\infty, \qquad y \mapsto y + \frac{\frac{1}{2}(\theta_\infty - \theta_0)}{x - \eta_\infty}. \tag{5.20}$$

Finally, we introduce the transformation ξ by $\xi = \chi\gamma\beta\gamma$, where β and γ are defined in section 2. This also survives in the reduction procedure to $P_{III'}$, and we get

$$\begin{aligned} \xi\colon \quad & \eta_0 \mapsto -\eta_\infty, \qquad \eta_\infty \mapsto -\eta_0, \qquad \theta_0 \mapsto \theta_\infty - 1, \qquad \theta_\infty \mapsto \theta_0 + 1, \\ & y \mapsto \frac{t}{y}, \qquad x \mapsto \frac{y}{t}\left[\frac{1}{2}(\theta_0 + \theta_\infty) - yx\right], \end{aligned} \tag{5.21}$$

and $\theta_- \mapsto -\theta_- + 1$.

Proposition 5.1 [10]**.** *The transformations s_1, s_2, ξ and $\psi(\lambda, \mu)$ generate the group of Bäcklund transformations of $P_{III'}$.*

Thus, we recover the (extended) affine Weyl group symmetry of $P_{III'}$ from the symmetry of Yang's equation. Note that one can put $\theta_\pm = (\theta_\infty \pm \theta_0)/2$.

Acknowledgment

The author would like to thank Professor H Kimura for his fruitful comments.

Appendix A. Yang's equation and the Ernst equation

Here, we give a brief remark on the reduction process from Yang's equation to the Ernst equation [11] that describes the stationary axisymmetric vacuum gravitational fields.

The Ernst equation is given by

$$\begin{aligned} &\tilde{f}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_\zeta^2\right)\tilde{f} - (\partial_\rho \tilde{f})^2 - (\partial_\zeta \tilde{f})^2 + (\partial_\rho \psi)^2 + (\partial_\zeta \psi)^2 = 0, \\ &\tilde{f}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_\zeta^2\right)\psi - 2(\partial_\rho \tilde{f})(\partial_\rho \psi) - 2(\partial_\zeta \tilde{f})(\partial_\zeta \psi) = 0. \end{aligned} \tag{A.1}$$

Introducing the matrix $\widehat{J}$ by

$$\widehat{J} = \frac{1}{\tilde{f}}\begin{pmatrix} 1 & \psi \\ \psi & \tilde{f}^2 + \psi^2 \end{pmatrix}, \tag{A.2}$$

we see that the Ernst equation can be written as

$$\partial_\rho(\rho \widehat{J}^{-1}\partial_\rho \widehat{J}) + \partial_\zeta(\rho \widehat{J}^{-1}\partial_\zeta \widehat{J}) = 0. \tag{A.3}$$

The Ernst equation (A.3) can be derived from Yang's equation by a dimensional reduction. In fact, introducing the new coordinates

$$\rho = \sqrt{w\tilde{w}}, \qquad \theta = \sqrt{\frac{w}{\tilde{w}}}, \qquad \zeta = \frac{1}{2}(z - \tilde{z}), \qquad t = \frac{1}{2}(z + \tilde{z}), \tag{A.4}$$

and assuming that the J-matrix depends on ρ and ζ, we see that $\widehat{J}(\rho, \zeta) = J(z, \tilde{z}, w, \tilde{w})$ with ${}^t\widehat{J} = \widehat{J}$ satisfies the Ernst equation (A.3).

Proposition A.1 [12, 13]**.** *Define the functions* $A^{(n)}$ *and* $\widetilde{A}^{(n)}$ *by*

$$A^{(n)} = \begin{vmatrix} u_0 & \mathrm{i}u_1 & \mathrm{i}^2 u_2 & \dots & \mathrm{i}^{n-1}u_{n-1} \\ \mathrm{i}u_1 & u_0 & \mathrm{i}u_1 & \dots & \mathrm{i}^{n-2}u_{n-2} \\ \mathrm{i}^2 u_2 & \mathrm{i}u_1 & u_0 & \dots & \mathrm{i}^{n-3}u_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{i}^{n-1}u_{n-1} & \mathrm{i}^{n-2}u_{n-2} & \mathrm{i}^{n-3}u_{n-3} & \dots & u_0 \end{vmatrix} \tag{A.5}$$

and $\widetilde{A}^{(n)} = A^{(n)}\begin{bmatrix} 1 \\ n \end{bmatrix}$*, respectively, where the entries* $u_j\,(j = 0, 1, 2, \dots)$ *satisfy*

$$\left(\partial_\rho - \frac{j}{\rho}\right)u_{j-1} = \partial_\zeta u_j, \qquad \left(\partial_\rho + \frac{j-1}{\rho}\right)u_j = -\partial_\zeta u_{j-1} \tag{A.6}$$

and the linear equation

$$\left(\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_\zeta^2 - \frac{j^2 - 1}{\rho^2}\right)u_j = 0. \tag{A.7}$$

Then

$$\widehat{J} = \frac{1}{A^{(n)}}\begin{pmatrix} \rho^{-n+1}A^{(n-1)} & \mathrm{i}\widetilde{A}^{(n+1)} \\ \mathrm{i}\widetilde{A}^{(n+1)} & \rho^{n-1}A^{(n+1)} \end{pmatrix} \tag{A.8}$$

or

$$\tilde{f} = \rho^{n-1}\frac{A^{(n)}}{A^{(n-1)}}, \qquad \psi = \mathrm{i}\rho^{n-1}\frac{\widetilde{A}^{(n+1)}}{A^{(n-1)}} \tag{A.9}$$

gives rise to a family of solutions to the Ernst equation.

Note that we have the bilinear relations

$$\begin{aligned} D_\rho A^{(n)}\cdot\widetilde{A}^{(n)} + \mathrm{i}D_\zeta\widetilde{A}^{(n+1)}\cdot A^{(n-1)} &= -\frac{n-2}{\rho}A^{(n)}\widetilde{A}^{(n)},\\ D_\rho\widetilde{A}^{(n+1)}\cdot A^{(n-1)} + \mathrm{i}D_\zeta A^{(n)}\cdot\widetilde{A}^{(n)} &= -\frac{n-1}{\rho}\widetilde{A}^{(n+1)}A^{(n-1)} \end{aligned} \tag{A.10}$$

and

$$A^{(n+1)}A^{(n-1)} = [A^{(n)}]^2 - [\widetilde{A}^{(n+1)}]^2. \tag{A.11}$$

Let us show that the above family of solutions to the Ernst equation can be obtained from that to Yang's equation given in proposition 2.2. We take the entries of determinant $\varphi_j\,(j\in\mathbb{Z})$ as

$$\varphi_j = \theta^j\rho^{-1}u_j, \qquad u_j = u_j(\rho,\zeta), \tag{A.12}$$

where ρ, ζ and θ are defined by (A.4). Then, the linear relation (2.10) and the Laplace equation (2.11) are reduced to (A.6) and (A.7), respectively. One can set $u_{-j} = (-1)^j u_j\,(j\in\mathbb{Z}_{\geqslant 0})$, which is consistent with (A.6) and (A.7). In this setting, we see that the functions τ_n^m defined in proposition 2.2 yield

$$\begin{aligned} \tau_n^0 &= (-1)^{\binom{n}{2}}\rho^{-n}A^{(n)},\\ \tau_n^1 &= (-1)^{\binom{n}{2}}\theta^n\rho^{-n}\mathrm{i}^{-n}\widetilde{A}^{(n+1)},\\ \tau_n^{-1} &= (-1)^{\binom{n+1}{2}}\theta^{-n}\rho^{-n}\mathrm{i}^{-n}\widetilde{A}^{(n+1)}, \end{aligned} \tag{A.13}$$

and the bilinear relations (2.13) are reduced to (A.10) and (A.11). A solution to Yang's equation

$$J = \frac{1}{\tau_n^0}\begin{pmatrix} \tau_{n-1}^0 & \tau_n^1 \\ -\tau_n^{-1} & -\tau_{n+1}^0 \end{pmatrix} \tag{A.14}$$

is reduced to

$$J = \frac{1}{A^{(n)}}\begin{pmatrix} (-1)^{n-1}\rho A^{(n-1)} & \theta^n\mathrm{i}^{-n}\widetilde{A}^{(n+1)} \\ -\theta^{-n}\mathrm{i}^n\widetilde{A}^{(n+1)} & (-1)^{n+1}\rho^{-1}A^{(n+1)} \end{pmatrix}. \tag{A.15}$$

Taking appropriate matrices $M = M(z,w)$ and $\widetilde{M} = \widetilde{M}(\tilde{z},\tilde{w})$, we see that $\widehat{J} = M^{-1}J\widetilde{M}$ coincides with (A.8).

Appendix B. From the Ernst equation to P_{III}

In this section, we mention the reduction from the Ernst equation to P_{III}.

Let us consider the equation

$$(\rho J^{-1}\partial_\rho J)_\rho + (\rho J^{-1}\partial_\zeta J)_\zeta = 0, \tag{B.1}$$

where $J = J(\rho,\zeta)$ is the matrix-valued function given by

$$J = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \qquad \mathcal{AD} - \mathcal{BC} = 1. \tag{B.2}$$

Obviously, equation (B.1) is an asymmetric version of the Ernst equation (A.3). Set

$$\begin{aligned} \mathcal{A}(\rho,\zeta) &= A(\rho), & \mathcal{B}(\rho,\zeta) &= B(\rho)\mathrm{e}^{\kappa\zeta}, \\ \mathcal{C}(\rho,\zeta) &= C(\rho)\mathrm{e}^{-\kappa\zeta}, & \mathcal{D}(\rho,\zeta) &= D(\rho). \end{aligned} \tag{B.3}$$

Then, equation (B.1) can be written as

$$\begin{aligned} &(A'D - BC')' = 0, \\ &(B'D - BD')' + \kappa^2\rho^2 BD = 0, \\ &(A'C - AC')' - \kappa^2\rho^2 AC = 0, \\ &(AD' - B'C)' = 0, \end{aligned} \tag{B.4}$$

with $AD - BC = 1$, where we denote $' = \rho\frac{\mathrm{d}}{\mathrm{d}\rho}$. By the similar procedure to that in section 5, one can obtain the Hamiltonian system for $\mathrm{P_{III}}$.

When we set $\mathcal{A} = \mathcal{D}$,

$$\widehat{\mathcal{J}} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} J \tag{B.5}$$

satisfies the Ernst equation (A.3). It is easy to see that the constraint $\mathcal{A} = \mathcal{D}$ yields the condition $\theta_\infty = \theta_0$ in the procedure to derive $\mathrm{P_{III}}$. This coincides with the result in [14].

Let us consider the specialization $u_j(\rho,\zeta) = (-\kappa)^j\rho\phi_j(\rho)\,\mathrm{e}^{\kappa\zeta}$, which is consistent with (B.3). Then, the linear relations (A.6) and (A.7) are reduced to

$$\left(\partial_\rho - \frac{j}{\rho}\right)\phi_j = -\kappa^2\phi_{j+1}, \qquad \left(\partial_\rho + \frac{j}{\rho}\right)\phi_j = \phi_{j-1} \tag{B.6}$$

and

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \kappa^2 - \frac{j^2}{\rho^2}\right)\phi_j = 0, \tag{B.7}$$

respectively. This means that the functions ϕ_j can be expressed in terms of the Bessel functions whose parameter is a non-negative integer. The functions $A^{(n)}$ and $\widetilde{A}^{(n+1)}$ are reduced to

$$\begin{aligned} A^{(n)} &= (-1)^{\binom{n}{2}}\kappa^{-2\binom{n}{2}}\rho^{n(2-n)}\,\mathrm{e}^{n\kappa\zeta}\sigma_n^0, \\ \widetilde{A}^{(n+1)} &= \mathrm{i}^n(-1)^{\binom{n+1}{2}}\kappa^{n(2-n)}\rho^{n(2-n)}\,\mathrm{e}^{n\kappa\zeta}\sigma_n^1, \end{aligned} \tag{B.8}$$

where σ_n^j is given by

$$\sigma_n^j = \begin{vmatrix} \phi_j^{(0)} & \phi_j^{(1)} & \cdots & \phi_j^{(n-1)} \\ \phi_j^{(1)} & \phi_j^{(2)} & \cdots & \phi_j^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_j^{(n-1)} & \phi_j^{(n)} & \cdots & \phi_j^{(2n-2)} \end{vmatrix}, \qquad \phi_j^{(k)} = \left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)^k\phi_j. \tag{$B.9$}$$

This leads us to the classical transcendental solutions of $\mathrm{P_{III}}$ given in proposition 4.1 with the additional constraint $\theta_0 = \theta_\infty$.

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